Supplemental Material for "High-precision force sensing using a single trapped ion"

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ELIMINATION OF THE VIBRATIONAL DEGREE OF FREEDOM

Let us make the canonical transformation of Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{\rm int}$,

$$\begin{aligned}\hat{H}_{\rm eff} &= e^{-\hat{S}}\hat{H}e^{\hat{S}} = \hat{H}_0 + \hat{H}_{\rm int} + [\hat{H}_0, \hat{S}] + [\hat{H}_{\rm int}, \hat{S}] \\ &\quad + \tfrac{1}{2}[[\hat{H}_0, \hat{S}], \hat{S}] + \tfrac{1}{2}[[\hat{H}_{\rm int}, \hat{S}], \hat{S}] + \ldots . \end{aligned} \tag{1}$$

Our goal is to choose $\hat{S}$ in a such a way that all terms of order g in $\hat{H}_{\rm eff}$ are canceled and the first term describing the spin-boson interaction is of order g^2/ω. If we determine $\hat{S}$ by the condition

$$\hat{H}_{\rm int} + [\hat{H}_0, \hat{S}] = 0, \tag{2}$$

then the effective Hamiltonian becomes

$$\hat{H}_{\rm eff} \approx \hat{H}_0 + \tfrac{1}{2}[\hat{H}_{\rm int}, \hat{S}]. \tag{3}$$

Let us consider the time-dependent operator $\hat{S}(t) = e^{{\rm i}\hat{H}_0 t/\hbar}\hat{S}e^{-{\rm i}\hat{H}_0 t/\hbar}$, which obeys the Heisenberg equation ${\rm i}\hbar\dot{\hat{S}}(t) = [\hat{S}(t), \hat{H}_0]$. Using Eq. (2) we arrive at the equation

$${\rm i}\hbar\dot{\hat{S}}(t) = \hat{H}_{\rm int}(t), \tag{4}$$

where $\hat{H}_{\rm int}(t) = e^{{\rm i}\hat{H}_0 t/\hbar}\hat{H}_{\rm int}e^{-{\rm i}\hat{H}_0 t/\hbar}$. Solving Eq. (4) we determine the desired operator $\hat{S}$.

Jaynes-Cummings model

We identify $\hat{H}_0 = \hbar\omega\hat{a}^\dagger\hat{a}$ and $\hat{H}_{\rm int} = \hbar g(\sigma^-\hat{a}^\dagger + \sigma^+\hat{a}) + \frac{z_{\rm ax}F}{2}(\hat{a}^\dagger + \hat{a})$. Using Eq. (4) we obtain

$$\hat{S} = \frac{g}{\omega}(\sigma^+\hat{a} - \sigma^-\hat{a}^\dagger) + \frac{z_{\rm ax}F}{2\hbar\omega}(\hat{a} - \hat{a}^\dagger), \tag{5}$$

which fulfills the condition (2). For the effective Hamiltonian we derive

$$\begin{aligned}\hat{H}_{\rm eff} &= \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\left(\Delta - \frac{g^2}{2\omega}\right)\sigma_z - \hbar\Omega_F\sigma_x \\ &\quad - \frac{\hbar g^2}{\omega}\sigma_z\hat{a}^\dagger\hat{a} - \frac{\hbar g^2}{2\omega} - \frac{z_{\rm ax}^2F^2}{4\hbar\omega} + \hat{H}', \end{aligned} \tag{6}$$

where $\Omega_F = g z_{\rm ax}F/2\hbar\omega$ is the Rabi frequency and $\hat{H}' = \frac{1}{3}[[\hat{H}_{\rm int}, \hat{S}], \hat{S}] + \ldots$ contains the higher-order terms in (1). We find

$$\begin{aligned}\frac{1}{3}[[\hat{H}_{\rm int}, \hat{S}], \hat{S}] &= \frac{2g^2 z_{\rm ax}F}{3\omega^2}\sigma_z(\hat{a}^\dagger + \hat{a}) - \frac{4\hbar g^3}{3\omega^2}(\sigma^-\hat{a}^\dagger + \sigma^+\hat{a}) \\ &\quad - \frac{4\hbar g^3}{3\omega^2}(\sigma^-\hat{a}^\dagger\hat{a}^\dagger\hat{a} + \sigma^+\hat{a}^\dagger\hat{a}\hat{a}). \end{aligned} \tag{7}$$

As long as $g/\omega \ll 1$ the higher-order terms can be neglected and thus the lowest-order effective Hamiltonian is given by Eq. (6).

Quantum Rabi Model

Here the interaction Hamiltonian is $\hat{H}_{\rm int}=\hbar g\sigma_x(\hat{a}^\dagger+\hat{a})+\frac{z_{\rm ax}F}{2}(\hat{a}^\dagger+\hat{a})$ and the canonical transformation is given by the operator

$$\hat{S}=\frac{g}{\omega}\sigma_x(\hat{a}-\hat{a}^\dagger)+\frac{z_{\rm ax}F}{2\hbar\omega}(\hat{a}-\hat{a}^\dagger). \tag{8}$$

The effective Hamiltonian is

$$\hat{H}_{\rm eff}=\hbar\omega\hat{a}^\dagger\hat{a}-2\hbar\Omega_F\sigma_x-\frac{\hbar g^2}{\omega}-\frac{(z_{\rm ax}F)^2}{4\hbar\omega}. \tag{9}$$

Remarkably, due to the equality $[[\hat{H}_{\rm int},\hat{S}],\hat{S}]=0$ all higher-order terms in Eq. (1) vanish.

Jahn-Teller Model

Following the same procedure we have

$$\begin{aligned}&\hat{H}_0=\hbar\omega(\hat{a}_x^\dagger\hat{a}_x+\hat{a}_y^\dagger\hat{a}_y),\\ &\hat{H}_{\rm int}=\hbar g\sigma_x(\hat{a}_x^\dagger+\hat{a}_x)+\hbar g\sigma_y(\hat{a}_y+\hat{a}_y)+\frac{z_{\rm t}F_x}{2}(\hat{a}_x^\dagger+\hat{a}_x)\\ &+\frac{z_{\rm t}F_y}{2}(\hat{a}_y^\dagger+\hat{a}_y).\end{aligned} \tag{10}$$

In this case the canonical transformation is represented by the operator

$$\begin{aligned}\hat{S}&=\frac{g}{\omega}\sigma_x(\hat{a}_x-\hat{a}_x^\dagger)+\frac{g}{\omega}\sigma_y(\hat{a}_y-\hat{a}_y^\dagger)+\frac{z_{\rm t}F_x}{2\hbar\omega}(\hat{a}_x-\hat{a}_x^\dagger)\\ &+\frac{z_{\rm t}F_y}{2\hbar\omega}(\hat{a}_y-\hat{a}_y^\dagger).\end{aligned} \tag{11}$$

Using Eq. (11) we obtain the following effective Hamiltonian

$$\begin{aligned}\hat{H}_{\rm eff}&=\hbar\omega(\hat{a}_x^\dagger\hat{a}_x+\hat{a}_y^\dagger\hat{a}_y)-\hbar\Omega_x\sigma_x-\hbar\Omega_y\sigma_y-2{\rm i}\frac{\hbar g^2}{\omega}\\ &\times\sigma_z(\hat{a}_x\hat{a}_y^\dagger-\hat{a}_x^\dagger\hat{a}_y)-\frac{2\hbar g^2}{\omega}-\frac{z_{\rm t}^2|\vec{F}_\perp|^2}{4\hbar\omega}+\hat{H}',\end{aligned} \tag{12}$$

where $\Omega_{x,y}=gz_{\rm t}F_{x,y}/\hbar\omega$ are the respective Rabi driving frequencies. The next higher-order terms in $\hat{H}'$ (12) are given by

$$\begin{aligned}\frac{1}{3}[[\hat{H}_{\rm int},\hat{S}],\hat{S}]&=2{\rm i}\frac{g^2z_{\rm t}F_x}{\omega^2}\sigma_z(\hat{a}_y^\dagger-\hat{a}_y)-2{\rm i}\frac{g^2z_{\rm t}F_y}{\omega^2}\sigma_z(\hat{a}_x^\dagger-\hat{a}_x)\\ &-\frac{4\hbar g^3}{\omega^2}\sigma_y\{(\hat{a}_y^\dagger+\hat{a}_y)(1+2\hat{n}_x)-2\hat{a}_x^{\dagger 2}\hat{a}_y\\ &-2\hat{a}_x^2\hat{a}_y^\dagger\}-\frac{4\hbar g^3}{\omega^2}\sigma_x\{(\hat{a}_x^\dagger+\hat{a}_x)(1+2\hat{n}_y)\\ &-2\hat{a}_y^{\dagger 2}\hat{a}_x-2\hat{a}_y^2\hat{a}_x^\dagger\}.\end{aligned} \tag{13}$$

DYNAMICAL DECOUPLING

Let us consider the Jaynes-Cummings quantum probe in the presence of additional strong carrier driving field

$$\hat{H}=\hat{H}_{\rm JC}+\hat{H}_F+\hat{H}_{\rm d},\quad \hat{H}_{\rm d}=\hbar\Omega(\sigma_+e^{{\rm i}\phi}+\sigma_-e^{-{\rm i}\phi}). \tag{14}$$

Here Ω is the Rabi frequency for resonant carrier transition and ϕ is the respective phase where we set $\phi=0$. The effect of the strong carrier excitation is to create dressed states defined by $\sigma_x\left|\pm\right\rangle=\pm\left|\pm\right\rangle$ which are separated by energy gap $\hbar\Omega$. These states are immune to dephasing caused by the thermal fluctuations. Indeed, expressing the effective Hamiltonian in the dressed state basis $\sigma_x\,|\pm\rangle = \pm\,|\pm\rangle$ we have

$$\hat{H}_{\rm eff}^{\rm JC} = \hbar(\Omega - \Omega_F)(|+\rangle\langle+| - |-\rangle\langle-|) - \frac{\hbar g^2}{\omega}(|+\rangle\langle-| + |-\rangle\langle+|)\hat{a}^\dagger\hat{a}. \tag{15}$$

In this picture the effect of the thermally induce fluctuations is to drive transition between $|+\rangle$ and $|-\rangle$ states. However, as long as the energy gap Ω is much higher than g^2/ω ($\Omega \gg g^2/\omega$) such transitions are highly supressed which allows to neglect the second term in (15) in the rotating-wave approximation. In order to remove the dependence of the Rabi frequency Ω in the measured signal we suggest to use a spin-echo technique. First the system is prepared in the state $|\uparrow\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ which evolves under the action of Hamiltonian (15) for a time period $t/2$ such that we have $|\psi(t/2)\rangle = (e^{-{\rm i}(\Omega-\Omega_F)t/2}|+\rangle + e^{{\rm i}(\Omega-\Omega_F)t/2}|-\rangle)/\sqrt{2}$. Then the phase of the strong carrier driving field is switch by π and the state evolves to $|\psi(t/2)\rangle \rightarrow |\psi(t)\rangle = (e^{{\rm i}\Omega_F t}|+\rangle + e^{-{\rm i}\Omega_F t}|-\rangle)/\sqrt{2}$. Final read-out of the ion states in the $|\uparrow\rangle$, $|\downarrow\rangle$ basis yield probability outcome $P_\uparrow(t) = \cos^2(\Omega_F t)$.